MEASURING THE PROBABILITY OF EVENTUALLY BEARING n LIVE BIRTHS: AN EXTENSION OF FERTILITY TABLES*

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As everyone must die someday, it is meaningless to ask the probability of eventually dying after a given age. However, in the case of birth, the situation is different. Since not all women will have even one birth, it is of interest to contemplate, for instance, how large is the probability that a woman will eventually bear a birth after a given age, or what are the probabilities that a woman who has already had two births will eventually bear two, three or any additional number of births in her remaining lifetime.

In this paper, a methodology of computing the probability that a woman of parity m at age x will eventually bear a total of n births during her lifetime will be discussed. The term birth here refers to live birth only.

Fertility Tables

Using the age-parity specific birth probability, it is easy to construct a fertility table for each order of birth. The *n*th order birth will be born only once, if ever, to the women of parity n-1; this is analogous to the situation of death in the construction of a life table. The moment she bears her *n*th child she "dies" from the cohort of n-1 parity.

In the construction of a set of fertility tables by order of birth, we consider a hypothetical cohort of women for each parity. For simplicity, we assume they live through their childbearing age. In a fertility table (see Table 1) we introduce the following symbols:

Column (1). x - Exact year of age of women as in a life table. In a fertility table, only childbearing age may be of concern.

Column (2). $w_{n-1,x}$ - This is the number of women of the cohort of parity n-1 at exact age x, corresponding to the l_x column in a life table. We consider a radix of 100,000 women at the beginning of the childbearing age for each parity. As the women who deliver their *n*th birth will become the women of parity n, these new mothers will leave the cohort. Thus, in each of the successive ages the cohort will be depleted by the number of births born in the previous age. (In actuality, the mothers of these births will become the new access to the parity of the next rank.)

Column (3). $f_{n,x}$ - This is the age-parity specific birth probability which is the backbone in preparing the fertility table as the probability of death in the life table. It is the probability that a woman of parity n-1 at exact age x will bear a child before reaching exact age x+1.[1] This specific measure of fertility is now available for the U. S.[2]

Column (4). $b_{n,x}$ - The number of the *n*th order births born to the women of the cohort of parity n-1 between ages x and x+1 when they are subjected to the corresponding age-parity specific birth probability given in Column (3). that is,

$$b_{n,x} = f_{n,x}\, w_{n-1,x} .$$

$$w_{n-1,x+1} = w_{n-1,x} - b_{n,x} .$$

Column (5). $B_{n,x}$ - This indicates the total number of the *n*th order births born to the women of the cohort of n-1 parity after age x. It is obtained by summing up b_n from the bottom of Column (4) successively till x, inclusive, i.e.,

$$B_{n,x} = \sum_{i=x} b_{n,i} .$$

Column (6). $F_{n,x}$ - This value shows the probability that a woman of parity n-1 at age x will eventually bear her *n*th child in her remaining lifetime. In short, it may be called the probability of the *n*th order birth, or the next birth, after age x to a woman of n-1 parity. The value is obtained by dividing $B_{n,x}$ by $w_{n-1,x}$.

An illustration is given in Table 1 for the first order birth based on the experience of the U. S. women in 1957. Obviously it is a period table.

There may be a few more functions in the fertility table if we wish to include the average year of sterility before the next birth, the counterpart in a life table being the life expectancy. These functions are not considered in the present paper since our interest centers in the probability of additional births after the mother reaches a certain age.

Measurement of the Probability of n Births

Thus, the probability that a woman of a given parity and a given age will eventually bear *an* additional child in her lifetime may be easily obtained. However, in actuality this is not the probability of bearing *exactly* one more

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birth but that of bearing *at least* one more birth, because some of the women who proceed to the parity of the next rank by having an additional child will bear further births.

Our next question is this: What is the probability that a woman of a given parity and age will eventually bear several additional births in her life? We introduce a notation ${}_nF_{m,x}$ to represent the probability that a woman of parity m at age x will eventually bear a total of at least n births in her lifetime; in short, we may call it the probability of n births after age x to a woman of m parity. Therefore, our notation $F_{n,x}$ in the above fertility table is, in fact, ${}_nF_{n-1,x}$. We assume that no two births can occur in a year to a woman.

Let us first consider the simplest case of ${}_2F_{0,x}$, i.e., the probability of having at least 2 births from a woman with no previous experience of childbirth at age x. Notice that:

$$1-f_{1,x} = \Pr\{\text{a woman of 0 parity and age x will not bear a child before she reaches age x+1}\}$$

$$= w_{0,x+1} / w_{0,x}$$

Now,

${}_2F_{0,x}$ = Pr{the first birth in (x, x+1) and the second birth any time after x+1} or Pr{no birth till x+1 and the first birth in (x+1, x+2) and the second birth any time after x+2} or Pr{no birth till x+2 and the first birth in (x+2, x+3) and the second birth any time after x+3} or · · ·

$$= f_{1,x} F_{2,x+1} + (1-f_{1,x}) f_{1,x+1} F_{2,x+2} + (1-f_{1,x}) (1-f_{1,x+1}) f_{1,x+2} F_{2,x+3} + \cdots$$

$$= \frac{b_{1,x}}{w_{0,x}} F_{2,x+1} + \frac{w_{0,x+1}}{w_{0,x}} \frac{b_{1,x+1}}{w_{0,x+1}} F_{2,x+2} + \frac{w_{0,x+1}}{w_{0,x}} \frac{w_{0,x+2}}{w_{0,x+1}} \frac{b_{1,x+2}}{w_{0,x+2}} F_{2,x+3} + \cdots$$

Therefore,

$$ {}_2F_{0,x} = \frac{1}{w_{0,x}} \sum_{i=x} b_{1,i} F_{2,i+1}. \qquad (1)$$

For the computation of ${}_3F_{0,x}$ we need to consider all possible paths that 3 births can take place through the childbearing age after x. Noticing that we already have $F_{3,x}$, our attention centers on the different paths taken by the first two births only. For a woman with no previous childbirth till age x, the earliest possible two births will occur in the manner of the first birth occurring during ages x and x+1 and the second birth during ages x+1 and x+2; there is only one possible path for such a sequence. Therefore, the probability is:

$$f_{1,x} f_{2,x+1} F_{3,x+2} = \frac{b_{1,x}}{w_{0,x}} \frac{b_{2,x+1}}{w_{1,x+1}} F_{3,x+2}.$$

There are two possible paths if the second birth is to occur during ages x+2 and x+3: (a) the first birth during x and x+1 and no birth during x+1 and x+2 and the second birth during x+2 and x+3 or (b) no birth till x+1 and two successive births in the following two years. The corresponding probabilities are:

$$f_{1,x} (1-f_{2,x+1}) f_{2,x+2} F_{3,x+3} = \frac{b_{1,x}}{w_{0,x}} \frac{b_{2,x+2}}{w_{1,x+1}} F_{3,x+3}.$$

and

$$(1-f_{1,x}) f_{1,x+1} f_{2,x+2} F_{3,x+3} = \frac{b_{1,x+1}}{w_{0,x}} \frac{b_{2,x+2}}{w_{1,x+2}} F_{3,x+3}.$$

In a similar manner, we see there are three different paths for the second birth taking place during x+3 and x+4 and the probabilities may be written as follows:

$$f_{1,x} (1-f_{2,x+1}) (1-f_{2,x+2}) f_{2,x+3} F_{3,x+4} = \frac{b_{1,x}}{w_{0,x}} \frac{b_{2,x+3}}{w_{1,x+1}} F_{3,x+4}.$$

$$(1-f_{1,x}) f_{1,x+1} (1-f_{2,x+2}) f_{2,x+3} F_{3,x+4} = \frac{b_{1,x+1}}{w_{0,x}} \frac{b_{2,x+3}}{w_{1,x+2}} F_{3,x+4}.$$

$$(1-f_{1,x}) (1-f_{1,x+1}) f_{1,x+2} f_{2,x+3} F_{3,x+4} = \frac{b_{1,x+2}}{w_{0,x}} \frac{b_{2,x+3}}{w_{1,x+3}} F_{3,x+4}.$$

There are 4 probabilities to be multiplied by $F_{3,x+5}$, in other words, 4 different paths the second birth may occur during x+4 and x+5. They are:

$$\frac{b_{1,x}}{w_{0,x}} \frac{b_{2,x+4}}{w_{1,x+1}}, \quad \frac{b_{1,x+1}}{w_{0,x}} \frac{b_{2,x+4}}{w_{1,x+2}}, \quad \frac{b_{1,x+2}}{w_{0,x}} \frac{b_{2,x+4}}{w_{1,x+3}}, \quad \frac{b_{1,x+3}}{w_{0,x}} \frac{b_{2,x+4}}{w_{1,x+4}}.$$

Therefore,

$$_3F_{0,x} = \frac{1}{w_{0,x}} [\frac{b_{1,x}}{w_{1,x+1}} (b_{2,x+1} F_{3,x+2} + b_{2,x+2} F_{3,x+3} + b_{2,x+3} F_{3,x+4} + \cdots) + \frac{b_{1,x+1}}{w_{1,x+2}} (b_{2,x+2} F_{3,x+3} + b_{2,x+3} F_{3,x+4} + \cdots) + \cdots]$$

$$= \frac{1}{w_{0,x}} [\frac{b_{1,x}}{w_{1,x+1}} (\sum_{j=x+1} b_{2,j} F_{3,j+1}) + \frac{b_{1,x+1}}{w_{1,x+2}} (\sum_{j=x+2} b_{2,j} F_{3,j+1}) + \cdots].$$

That is,

$$_3F_{0,x} = \frac{1}{w_{0,x}} [\sum_{i=x} \frac{b_{1,i}}{w_{1,i+1}} (\sum_{j=i+1} b_{2,j}F_{3,j+1})]. \quad (2)$$

In a similar manner, it can be shown that:

$$_4F_{0,x} = \frac{1}{w_{0,x}} [\sum_{i=x} \frac{b_{1,i}}{w_{1,i+1}} \{\sum_{j=i+1} \frac{b_{2,j}}{w_{2,j+1}} (\sum_{k=j+1} b_{3,k} F_{4,k+1})\}]. \quad (3)$$

In general,

$$_nF_{0,x} = \frac{1}{w_{0,x}} [\sum_{i=x} \frac{b_{1,i}}{w_{1,i+1}} \{\sum_{j=i+1} \frac{b_{2,j}}{w_{2,j+1}} (\cdots \frac{b_{n-2,p}}{w_{n-2,p+1}} [\sum_{q=p+1} b_{n-1,q} F_{n,q+1}])\}]. \quad (4)$$

and

$$_nF_{m,x} = \frac{1}{w_{m,x}} [\sum_{i=x} \frac{b_{m+1,i}}{w_{m+1,i+1}} \{\sum_{j=i+1} \frac{b_{m+2,j}}{w_{m+2,j+1}} \cdots (\sum_{q=p+1} b_{n-1,q} F_{n,q+1})\}]. \quad (5)$$

Obviously, the probability that a woman of parity m at age x will bear *exactly* a total of n children in her lifetime will be given by $_nF_{m,x} - {}_{n+1}F_{m,x}$. If we denote the probability of at least n births at the beginning of the childbearing age by F_n, the theoretical distribution of the women of completed fertility may be easily computed. The proportion of the childless women will be given by $1-F_1$ and the proportion of n births by $F_n - F_{n+1}$.

Application

Material to illustrate the procedure above are provided by a publication of P. K. Whelpton and A. A. Campbell.[3] The age-parity specific birth probability was calculated for the calendar year of 1957 when the U. S. women showed the highest fertility rate in recent years. In their publication, P. K. Whelpton and A. A. Campbell presented the central and the cumulative birth rates by order of birth and by age for each birth cohort of women. The cumulative rate enables one to derive age-parity specific birth probabilities.

Let $Q_{i,x}(y)$ be the cumulative birth rate of the *i*th order birth by a woman reaching age x on January 1 of the calendar year y. Then the age-parity specific birth probability for age x, parity i-1, and calendar year y, $f_{i,x}(y)$, may be obtained as follows:

$$f_{i,x}(y) = \frac{Q_{i,x+1}(y+1) - Q_{i,x}(v)}{Q_{i-1,x}(y) - Q_{i,x}(y)}$$

In particular,

$$f_{1,x}(y) = \frac{Q_{1,x+1}(y+1) - Q_{1,x}(y)}{1-Q_{1,x}(y)}$$

This appears to be consistent with the method used by the National Center for Health Statistics in computing the age-parity specific birth probability.[4]

With these age-parity specific birth probabilities thus obtained and certain fertility functions shown in Table 1, $_nF_{m,x}$ may be computed. As n becomes larger in comparison with m, the procedure becomes progressively tedious. As an example, the computational procedure of $_3F_{0,x}$ is illustrated in Table 2 and the explanation for each column of the Table is given below.

Column (1). x - Exact year of age of women as in Table 1.

Column (2). $b_{2,x+1}$ - The number of the second order births born to women of the parity-one-cohort between ages x+1 and x+2. The entry of this column is transcribed from Column (4) of an appropriate fertility table for the second order birth such as Table 1 shifting a row upward.

Column (3). $F_{3,x+2}$ - The probability of the third order birth after age x+2 for the woman of parity 2. It is transcribed from Column (6) of an appropriate fertility table for the third order birth shifting two rows upward.

Column (4). $b_{2,x+1}\ F_{3,x+2}$ - The product of the preceding two columns for the row of x. (This shows the number of the third order births eventually born after age x+2 if there are $b_{2,x+1}$ women of parity 2 at age x+2 and they are exposed to the given series of $f_{3,x}$ afterwards.)

Column (5). $\Sigma\ b_2\ F_3$ - The value obtained by adding the values in Column (4) from the bottom till x, inclusive. (It gives the total number of the third order births eventually born to all the women joining parity 2 from the parity-1-cohort in successive ages x+2 on. As evident, the number of women joining to parity 2 at age x is given by $b_{2,x-1}$.)

Column (6). $b_{1,x}$ - The number of the first order births born to the women of the parity-0-cohort between ages x and x+1. The entry of this column is transcribed from Column (4) of an appropriate fertility table for the first order birth such as Table 1.

Column (7). $w_{1,x+1}$ - The number of women of the parity-1-cohort at age x+1. It is transcribed from an appropriate fertility table shifting a row upward.

Column (8). $b_{1,x}\ /\ w_{1,x+1}$ - The quotient of the two preceding columns for the row of x. (It is the ratio of the number of women who would have joined to parity 1 from the parity-0-cohort at age x+1 to the number of women remaining in the parity-1-cohort at that age.)

Column (9). $\frac{b_1}{w_1}\ \Sigma\ b_2\ F_3$ - The product of Columns (5) and (8). (It represents the number of the third order births which would have eventually been born to $b_{1,x}$ women joining to parity 1 from the parity-0-cohort at age x+1.)

Column (10). $\Sigma\ \frac{b_1}{w_1}\ \Sigma\ b_2\ F_3$ - The sum of the values of Column (9) from the bottom till x. (The value shows the total number of the third order births eventually born to $w_{0,x}$ women of the parity-0-cohort at age x.)

Column (11). $w_{0,x}$ - The number of the women of the parity-0-cohort at age x, transcribed from Column (2) of an appropriate fertility table for the first order birth.

Column (12). ${}_3F_{0,x}$ - The quotient of the preceding two columns for the row of x. It is the probability that at least three births will eventually be born to a woman of 0 parity at age x when she is exposed to the given age-parity specific birth probabilities.

The attached figures show some of the curves of ${}_nF_{m,x}$ computed from the age-parity specific birth probabilities of the U. S. women experienced in 1957.

Summary

This paper is intended to present a method of measuring the probability that a woman of a given parity at age x will eventually have at least n additional births in her remaining lifetime.

A hypothetical cohort of women of equal radix for each parity is considered and the number of women of n parity at age x is denoted by $w_{n,x}$. We assume no mortality.

Define:

$f_{n,x}$ = Pr{a woman of n-1 parity at age x will bear her *n*th live birth between ages x and x+1}.

$b_{n,x}$ = the number of the *n*th order children born to the women between ages x and x+1

$= f_{n,x}\ w_{n-1,x}.$

Clearly,

$$w_{n-1,x+1} = w_{n-1,x} - b_{n,x}.$$

and

$(1-f_{n,x})$ = Pr{a woman of n-1 parity at age x will be sterile until age x+1}

$= w_{n-1,x+1}\ /\ w_{n-1,x}.$

It can easily be seen that the probability $F_{n,x}$ that a woman of n-1 parity at age x will eventually bear the next birth in her life is as follows:

$$F_{n,x} = \sum_{i=x} b_{n,i}\ /\ w_{n-1,x}.$$

The above functions, i.e., $w_{n,x}$, $f_{n,x}$, $b_{n,x}$ and $F_{n,x}$, may be arrayed in the form of a life table.

To find the probability ${}_nF_{m,x}$, that a woman of m parity at age x will eventually bear a total of at least n live births, let us first consider:

$${}_2F_{0,x} = f_{1,x}\ F_{2,x+1} + (1-f_{1,x})\ f_{1,x+1}\ F_{2,x+2} + (1-f_{1,x})\ (1-f_{1,x+1})\ f_{1,x+2}\ F_{2,x+3} + \cdots$$

$$= \frac{1}{w_{0,x}} \sum_{i=x} b_{1,i}\ F_{2,i+1}.$$

Similarly,

$${}_3F_{0,x} = \frac{1}{w_{0,x}} \sum_{i=x} \frac{b_{1,i}}{w_{1,i+1}} \left(\sum_{j=i+1} b_{2,j}\ F_{3,j+1}\right).$$

and

$$ {}_{n}F_{0,x} = \frac{1}{w_{0,x}} \sum_{i=x} \frac{b_{1,i}}{w_{1,i+1}} \Big[\sum_{j=i+1} \frac{b_{2,j}}{w_{2,j+1}} \Big\{ \Sigma \cdots $$

$$ \Big(\sum_{q=p+1} b_{n-1,q} \; F_{n,q+1} \Big) \Big\} \Big]. $$

In general,

$$ {}_{n}F_{m,x} = \frac{1}{w_{m,x}} \sum_{i=x} \frac{b_{m+1,i}}{w_{m+1,i+1}} \Big[\sum_{j=i+1} \frac{b_{m+2,j}}{w_{m+2,j+1}} $$

$$ \Big\{ \sum_{k=j+1} \cdots \Big(\sum_{q=p+1} b_{n-1,q} \; F_{n,q+1} \Big) \Big\} \Big]. $$

To illustrate, some ${}_{n}F_{m,x}$ curves are given based on the U. S. women experience in 1957, when the fertility rate was the highest in the post-war period.

References

1 National Center for Health Statistics, *Fertility Measurement: A Report of the United States National Committee on Vital and Health Statistics*. U. S. Public Health Service, Publication No. 1000, Series 4, No. 1, Washington, D. C., Nov. 1965.

2 See *Vital Statistics of the United States*, Volume 1, U. S. Public Health Service.

3 Whelpton, P. K. and Campbell, A. A. "Fertility tables for birth cohorts of American Women, part 1" *Vital Statistics--Special Reports*, Vol. 51, No. 1, Public Health Service, Washington, D. C., Jan. 1960.

4 Campbell, A. A. Personal Communication, March 21, 1967.

TABLE 1

FERTILITY TABLE FOR THE FIRST ORDER BIRTH, U. S. WOMEN, 1957

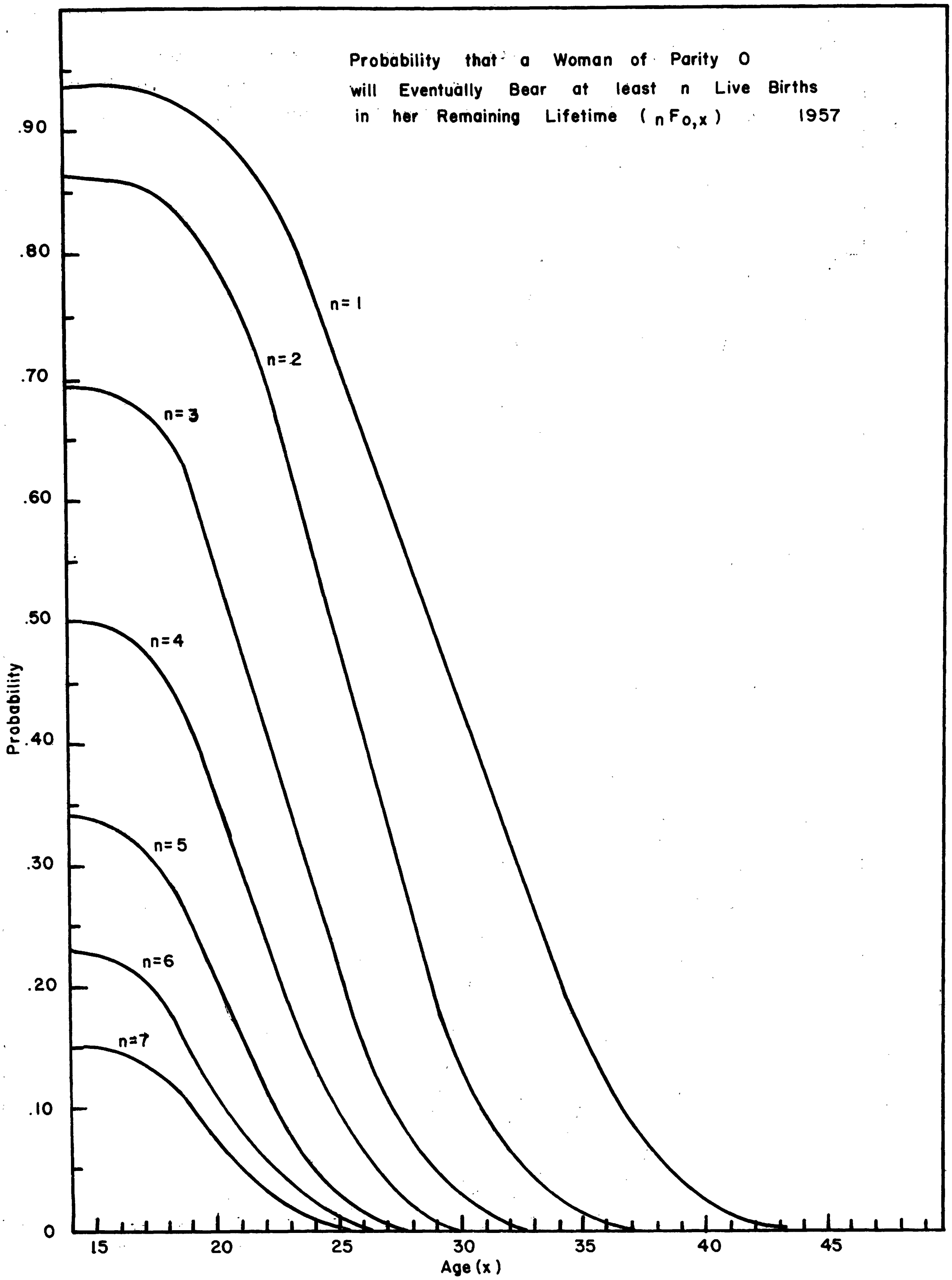
Age	No. of women of 0 parity at age x	Birth probability of the 1st order birth in age x	No. of the 1st order births born	Total no. of 1st order births born after age x	Probability of the 1st order birth after age x
x	$w_{0,x}$	$f_{1,x}$	$b_{1,x}$	$B_{1,x}$	$F_{1,x}$
(1)	(2)	(3)	(4)	(5)	(6)
14	100,000	.0048	480	93,787	.9379
15	99,520	.0141	1,403	93,307	.9376
16	98,117	.0390	3,826	91,904	.9367
17	94,291	.0728	6,864	88,078	.9341
18	87,427	.1116	9,757	81,214	.9289
19	77,670	.1575	12,233	71,457	.9200
20	65,437	.1782	11,661	59,224	.9050
21	53,776	.1951	10,492	47,563	.8845
22	43,284	.2059	8,912	37,071	.8565
23	34,372	.2077	7,139	28,159	.8192
24	27,233	.1971	5,363	21,020	.7719
25	21,865	.1730	3,783	15,652	.7158
26	18,082	.1479	2,674	11,869	.6564
27	15,408	.1361	2,097	9,195	.5968
28	13,311	.1172	1,560	7,098	.5332
29	11,751	.1030	1,210	5,538	.4713
30	10,541	.0853	899	4,328	.4106
31	9,642	.0738	712	3,429	.3556
32	8,930	.0682	609	2,717	.3043
33	8,321	.0577	480	2,108	.2533
34	7,841	.0494	387	1,628	.2076
35	7,454	.0463	345	1,241	.1665
36	7,109	.0376	267	896	.1260
37	6,842	.0342	234	629	.0919
38	6,608	.0224	148	395	.0598
39	6,460	.0132	85	247	.0382
40	6,375	.0097	62	162	.0254
41	6,313	.0066	42	100	.0158
42	6,271	.0039	24	58	.0092
43	6,247	.0025	16	34	.0054
44	6,231	.0015	9	18	.0029
45	6,222	.0008	5	9	.0014
46	6,217	.0003	2	4	.0006
47	6,215	.0002	1	2	.0003
48	6,214	.0001	1	1	.0002
49	6,213	.0001	0	0	.0001
50	6,213	.0000	0	0	.0000

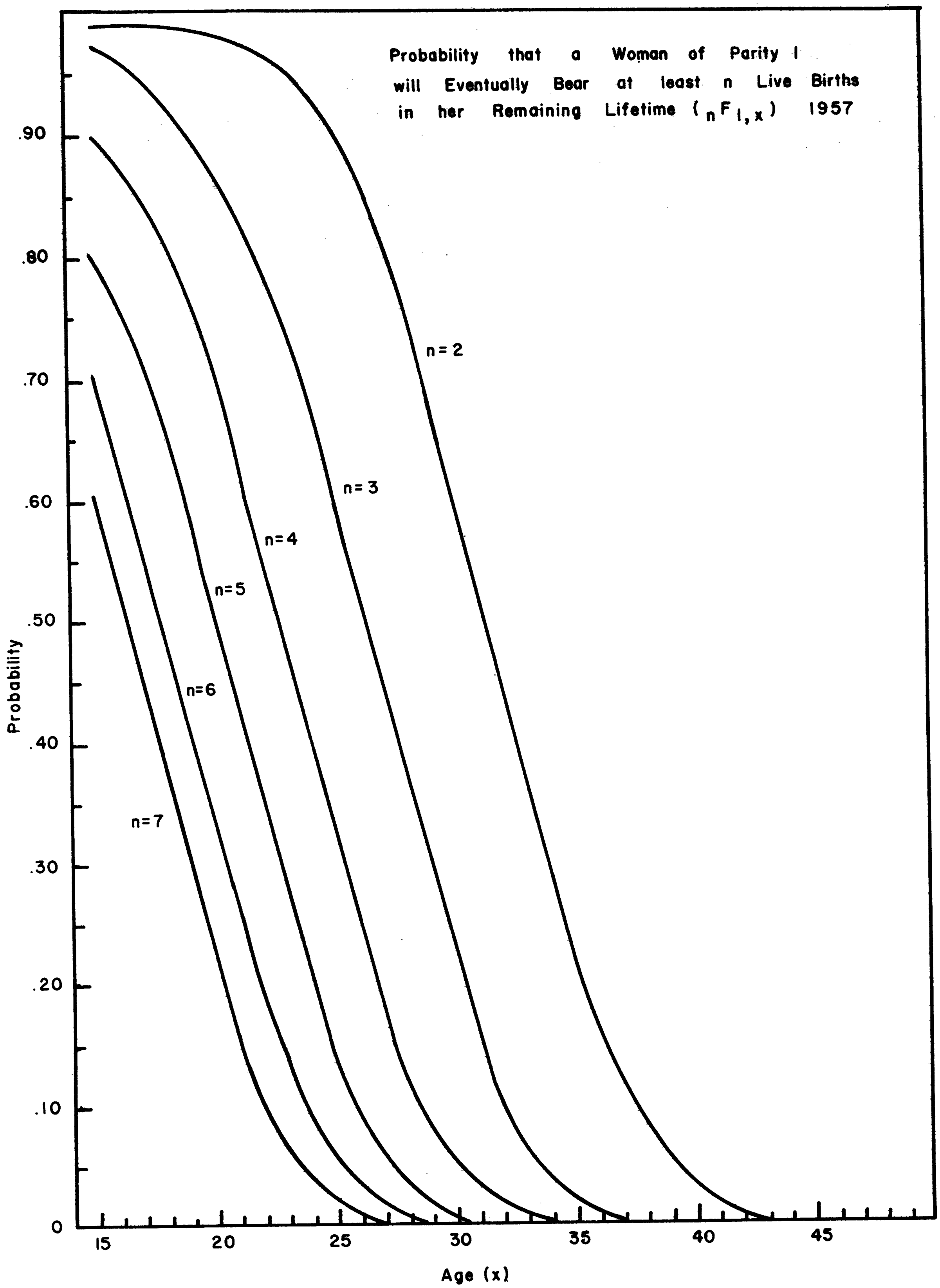
TABLE 2

CALCULATION OF ${}_3F_{0,x}$ — U. S. WOMEN, 1957

$${}_3F_{0,x} = \frac{1}{w_{0,x}} \left[\sum_{i=x} \frac{b_{1,i}}{w_{1,i+1}} \left(\sum_{j=i+1} b_{2,j}\, F_{3,j+1} \right) \right]$$

X	$b_{2,x+1}$	$F_{3,x+2}$	$b_{2,x+1}F_{3,x+2}$	$\Sigma\, b_2F_3$	$b_{1,x}$	$w_{1,x+1}$	$\frac{b_{1,x}}{w_{1,x+1}}$	$\frac{b_1}{w_1}\,\Sigma\, b_2F_3$	$\Sigma\,\frac{b_1}{w_1}\,\Sigma\, b_2F_3$	$w_{0,x}$	${}_3F_{0,x}$
(1)	(2)	(3)	(4)=(2)(3)	(5)=Σ(4)	(6)	(7)	(8)=(6)/(7)	(9)=(5)(8)	(10)=Σ(9)	(11)	(12)=(10)/(11)
14	26,000	.9950	25871.0400	97313.2190	480	100,000	.0048	467.1034	69298.8634	100,000	.6930
15	21,445	.9919	21272.1533	71442.1990	1403	74,000	.0190	1354.5437	68831.7600	99,520	.6916
16	17,380	.9875	17162.7500	50170.0257	3826	52,555	.0728	3652.3779	67477.2163	98,117	.6877
⋮											
38	5	.0474	.2371	.3715	148	180	.8222	.3054	.3917	6,608	.0001
39	3	.0293	.0880	.1344	85	175	.4857	.0653	.0863	6,460	.0000
40	2	.0176	.0317	.0464	62	172	.3605	.0167	.0210	6,375	.0000
41	1	.0098	.0108	.0146	42	170	.2471	.0036	.0043	6,313	.0000
42	1	.0048	.0029	.0039	24	169	.1420	.0006	.0007	6,271	.0000
43	0	.0024	.0007	.0010	16	168	.0952	.0001	.0001	6,247	.0000
⋮											


Probability that a Woman of Parity 0
will Eventually Bear at least n Live Births
in her Remaining Lifetime ($_nF_{0,x}$) 1957
n= 1
n=2
n= 3
n=4
n= 5
n=6
n=7
Probability
.90
.80
.70
.60
.50
.40
.30
.20
.10
0
15
20
25
30
35
40
45
Age (x)



Probability that a Woman of Parity 1 will Eventually Bear at least n Live Births in her Remaining Lifetime ($_nF_{1,x}$) 1957
n=2
n=3
n=4
n=5
n=6
n=7
.90
.80
.70
.60
.50
.40
.30
.20
.10
0
Probability
15
20
25
30
35
40
45
Age (x)